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13. ABSTRACT (Maximum 200 words)

We started out with the goal of un... ...ng approximation order in a multivariate context, including the approximation o... ...es. In addition, we wanted to understand better the use and analysis of our approa... ...multivariate polynomial interpolation.

We ended up concentrating on approxi...ation from *shift-invariant* spaces of functions on $\mathbb{R}^d$. Here, $\mathcal{S}$ is shift-invariant if $f \in \mathcal{S}$ implies that also $f(\cdot - \alpha) \in \mathcal{S}$ for any integer vector α. The simple, yet widely applicable, model we considered concerns the behavior, as $h \to 0$, of the distance $\operatorname{dist}(f, \mathcal{S}_h^h)$ of a (suitably smooth) f from the scaled space $\mathcal{S}_h^h := \{f(\cdot/h) : f \in \mathcal{S}_h\}$, with each $\mathcal{S}_h$ a shift-invariant space. Examples of such spaces are provided by finite elements on a regular grid, in particular, box spline spaces, also the spaces which make up the multiresolution analysis generated by *wavelets*.

14. SUBJECT TERMS

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Multivariate Spline Approximation

Final Report

Carl de Boor & Amos Ron
December 20, 1990

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U. S. Army Research Office
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Center for the Mathematical Sciences
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A. STATEMENT OF THE PROBLEM STUDIED

We started out with the goal of understanding approximation order in a multivariate context, including the approximation of surfaces. In addition, we wanted to understand better the use and analysis of our approach to multivariate polynomial interpolation.

We ended up concentrating on approximation from *shift-invariant* spaces of functions on $\mathbb{R}^d$. Here, $\mathcal{S}$ is **shift-invariant** if $f \in \mathcal{S}$ implies that also $f(\cdot - \alpha) \in \mathcal{S}$ for any integer vector α. The simple, yet widely applicable, model we considered concerns the behavior, as $h \to 0$, of the distance $\operatorname{dist}(f, S_h^h)$ of a (suitably smooth) f from the scaled space $\mathcal{S}_h^h := \{f(\cdot/h) : f \in \mathcal{S}_h\}$, with each $\mathcal{S}_h$ a shift-invariant space. Examples of such spaces are provided by finite elements on a regular grid, in particular, box spline spaces; also the spaces which make up the multiresolution analysis generated by *wavelets* are (scaled versions of) shift-invariant spaces, as are the related spaces generated by hierarchical bases (of use in the efficient numerical solution of elliptic PDEs); in the latter two examples, $h = 2^{-k}$, $k = 0, 1, 2, \ldots$. Finally, shift-invariant spaces had been recently invoked (in a quite unexpected way) for the construction of approximation schemes to *scattered* data (in 2-, 3-, and higher dimensions) by translates of radial basis functions.

B. SUMMARY OF THE MOST IMPORTANT RESULTS

This brief outline relies on the fact that more details can be found in the semi-annual reports submitted during the grant period, and, if need be, in the manuscripts filed with ARO as required. All boldfaced numbers refer to items in the list of publications given in **C**.

The highlight of the research done under this grant concerns approximation from *shift-invariant* spaces of functions on $\mathbb{R}^d$. In joint work with Ronald A. DeVore (whose semester-long visit was partially supported by this grant), we were able to obtain a complete characterization of the approximation order (in the $L_2(\mathbb{R}^d)$-norm) from a sequence $(\mathcal{S}_h^h)$ in case each $\mathcal{S}_h$ is a **principal** shift-invariant space, i.e., the L_2-closure $\mathcal{S}(\phi_h)$ of the finite linear combinations of shifts (= integer translates) of one function, ϕ_h. This characterization contains the socalled Strang-Fix conditions, which have dominated such considerations for the last twenty years, as a very special case. But, in contrast to the Strang-Fix conditions, our characterization is derived under no assumptions on the generating functions ϕ_h other than the obvious one that each should be an L_2-function. It is quite surprising that it was possible to solve this problem in this complete generality.

The solution (in **5**) is based on a new formula for the orthogonal projection onto a closed principal shift-invariant subspace of L_2 and on the resulting characterization of the elements of such a space. Both are in terms of the Fourier transform, as is the characterization of approximation order. Moreover, we found that the approximation order associated with a sequence (S_h) of arbitrary shift-invariant spaces is already realised by some sequence (S'_h) of *principal* shift-invariant spaces with $S'_h \subset S_h$. This settles (see **6**) a discussion which started when Strang and Fix asserted such a theorem for finitely generated shift-invariant spaces (in a certain restricted context) which was subsequently proven to be false (by Jia, while a student of de Boor) as stated, and whose correct

formulation has been the subject of several papers over the years. We know this result to be also of interest in the current attempts to construct 'short' wavelets.

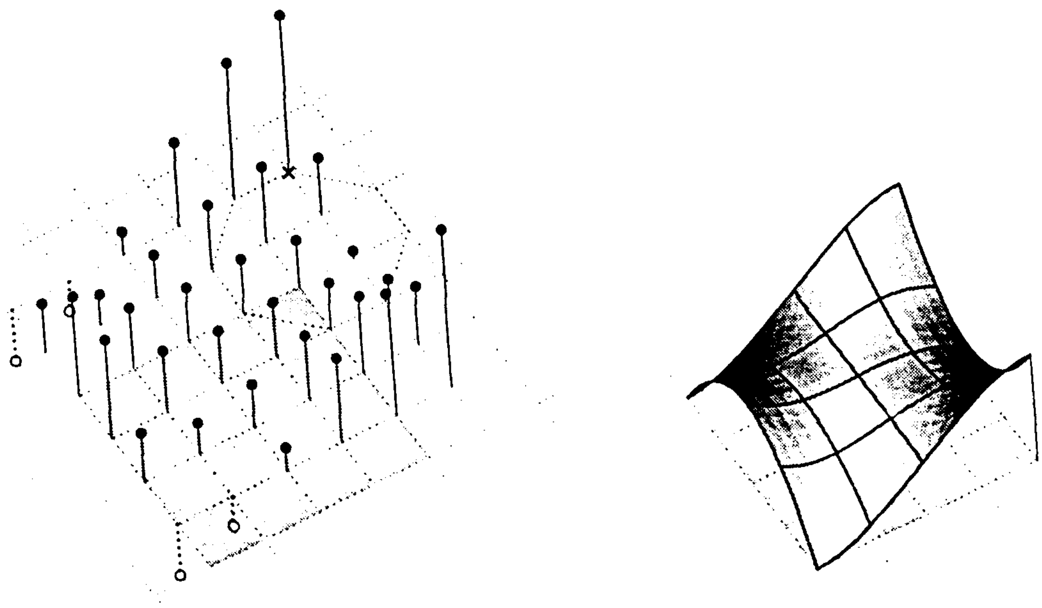
The results (in **6**) on the structure of finitely generated shift-invariant spaces provide a very convenient platform on which Amos Ron and Zuowei Shen are presently completing a (mathematical) machine for the analysis of frames and Riesz bases for the space $L_2(\mathbb{R}^d)$. That work focuses on the following practically important special cases: Weyl-Heisenberg frames (also known as Gabor frames, and which contain the short-window Fourier transform as a special case), and affine (i.e., wavelet) frames. One of the main achievements in this on-going work is a new and efficient way for computing the frame bounds of such sets. Also, new ways for calculating the dual frame are obtained. (To recall, efficient calculation of the dual frame is essential for any practical use of a frame.)

Shift-invariant spaces play a fundamental role in *wavelet theory*, particularly in the decomposition of $L_2(\mathbb{R}^d)$ via multiresolution analysis. Such a multiresolution analysis is given by a nested sequence $(\mathcal{S}^k)_{k\in\mathbb{Z}}$, with $\mathcal{S}^k$ generated by the $2^{-k}\mathbb{Z}^d$ shifts of some function φ_k, provided certain conditions are satisfied. Among these are (i) the density of the union of the $\mathcal{S}^k$ and (ii) the triviality of their intersection. Simple necessary and sufficient conditions for (i) and for (ii) are derived in **7** . In addition, **7** provides a new way to construct such multiresolution analysis and the corresponding wavelet spaces in the full multidimensional setting. In particular, we are able to deal with the situation when the spaces $\mathcal{S}^k$ are not all given in terms of the scales of just one function.

We are also very pleased about our results about approximation order from shift-invariant spaces when the error is measured in the *max-norm*. These results are all based on the observation in **15** that the approximation $\sum_{\alpha\in\mathbb{Z}^d}\phi(\cdot-\alpha)\exp(i\theta\alpha)$ from $\mathcal{S}(\phi)$ to $\exp(i\theta\cdot)\sum_{\alpha\in\mathbb{Z}^d}\exp(i\theta\alpha)$ is, within a θ-independent factor, as small as the best-possible error (as measured in the max-norm). This leads to the easy use of such bounded exponentials where traditionally polynomials were used for judging approximation order, thus permitting consideration (see **10**) of 'basic' functions ϕ which are not compactly supported or decay fast at infinity. In particular, surprising results concerning approximation with radial basis functions are obtained.

The basic results of **5** and **10** have already been invoked by several different groups for the study of certain special cases. As far as our own work is concerned, we made applications of the results of **5** to non-stationary multiresolution analysis in **13** and to radial basis functions in **17** . Also, the scheme of **10** was studied in norms other than the max-norm in **19** .

While the book **9** ,"Box Splines", by C. de Boor, K. Höllig, and S. Riemenschneider, was described as close to completion in the Final Report for a previous grant, the actual finishing turned out to take considerably more time. This was not only due to difficulties with the publishers (who decided to change the agreed-upon format after delivery of the final, camera-ready manuscript on 1 September 1992). Rather, it seemed opportune to include in the book recent results on the use of box splines in the construction of multivariate *wavelets*. Also, the Notes at the end of each chapter, concerning the existing literature, turned out to require much more time and care than anticipated. Finally, it seemed good to carry out an investigation, see **4** , into the computational aspects of box splines before publishing the book. Here is one of the 57 illustrations in the book.



(1)**Figure.** Part of a bivariate cardinal spline (based on the ZP element) and the relevant box spline coefficients, i.e., those coefficients for which the support of the corresponding shifted box spline overlaps the domain on which the spline is plotted.

C. LIST OF ALL PUBLICATIONS AND TECHNICAL REPORTS

1. C. de Boor, "On the error in multivariate polynomial interpolation", *Applied Numerical Mathematics* **10** (1992), 297–305.
2. C. de Boor, "Approximation order without quasi-interpolants", in *Approximation Theory VII* (E. W. Cheney, C. Chui, and L. Schumaker, eds), Academic Press (New York), (1992), 1–18.
3. C. de Boor, "Multivariate piecewise polynomials", *Acta Numerica* (1993), 65–109.
4. C. de Boor, "On the evaluation of box splines", *Advances in Comp. Math.* **xx** (1993), xxx–xxx.
5. C. de Boor, R. DeVore and A. Ron, "Approximation from shift-invariant subspaces of $L_2(\mathbb{R}^d)$", *Trans. Amer. Math. Soc.* **xx** (1994), xxx–xxx.
6. C. de Boor, R. DeVore and A. Ron, "The structure of finitely generated shift-invariant spaces in $L_2(\mathbb{R}^d)$", *J. Funct. Anal.* **xx** (199x), xxx–xxx.
7. C. de Boor, R. DeVore and A. Ron, "On the construction of multivariate (pre)-wavelets", *Constr. Approx.* **9** (1993), 123–166.
8. C. de Boor, K. Höllig and S. D. Riemenschneider, *Box Splines*, xvii + 200p, Springer-Verlag (New York), 1993.

9. C. de Boor and Rong-Qing Jia, "A sharp upper bound on the approximation order of smooth bivariate pp functions", *J. Approx. Theory* **72(1)** (1993), 24–33.

10. C. de Boor and A. Ron, "Fourier analysis of the approximation power of principal shift-invariant spaces", *Constr. Approx.* **8** (1992), 427–462.

11. C. de Boor, Amos Ron and Zuowei Shen, "On ascertaining inductively the dimension of the joint kernel of certain commuting linear operators", *Advances in Appl. Math.* **xx** (1994), xxx–xxx.

12. N. Dyn, I. R. H. Jackson, D. Levin and A. Ron, "On multivariate approximation by the integer translates of a basis function", *Israel J. Math.* **78** (1992), 95–130.

13. N. Dyn and A. Ron, "Multiresolution analysis by infinitely differentiable compactly supported functions", CMS TSR #93-4, U.Wisconsin-Madison, (1993).

14. N. Dyn and A. Ron, "Radial basis function approximation: from gridded centers to scattered centers", CMS TSR #94-3, December, (1993).

15. A. Ron, "A characterization of the approximation order of multivariate spline spaces", *Studia Math.* **98(1)** (1991), 73–90.

16. A. Ron, "Remarks on the linear independence of the integer translates of exponential box splines", *J. Approx. Theory* **71(1)** (1992), 61–66.

17. A. Ron, "The L_2-approximation orders of principal shift-invariant spaces generated by a radial basis function", in *Numerical Methods in Approximation Theory, ISNM 105* (D. Braess, L.L. Schumaker, ed), Birkhäuser (Basel), (1992), 245–268.

18. A Ron, "Negative observations concerning approximations from spaces generated by scattered shifts of functions vanishing at ∞", *J. Approx. Theory* **xx** (1994), xxx–xxx.

19. A. Ron, "Approximation orders of, and approximation maps from, local principal shift-invariant spaces", Submitted to J. Approx. Theory , (1993).

20. A. Ron, "Characterizations of linear independence and stability of the shifts of a univariate refinable function in terms of its refinement mask", CMS TSR #93-3, U.Wisconsin-Madison, (1993).

21. A. Ron and N. Sivakumar, "The approximation order of box spline spaces", *Proc. Amer. Math. Soc.* **117** (1993), 473–482.

D. LIST OF ALL PARTICIPATING SCIENTIFIC PERSONNEL SHOWING ANY ADVANCED DEGREES EARNED BY THEM WHILE EMPLOYED ON THE PROJECT

C. de Boor (entire time), A. Ron (entire time), Ronald A. DeVore (partial support during 15 February - 15 May, 1991), Nira Dyn (partial support during July - August 1991), Also, a student, Scott Kersey, was supported 40% during the academic year 1992/93.

In addition, a student, Kang Zhao, not supported by the grant completed a thesis (and several papers) concerning approximation from shift-invariant spaces. Also, a postdoc, Zuowei Shen, while not supported by this grant, cooperated with C. de Boor and Amos Ron on several papers, and this cooperation is on-going.